FAULT DETECTION BASED ON MODIFIED INTERVAL OBSERVER FOR DISCRETE-TIME LINEAR SYSTEM

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ABSTRACT: In the paper, a fault detection scheme based on modified interval observer is proposed for discrete-time linear system. Firstly, a modified interval observer, which has more design degrees of the freedom is applied to fault detection. In addition, the interval observer has a natural threshold, which can omit the two links of designing the residuals evaluator and threshold selector. Then, we introduce $l_1/H_{\infty}$ performance for the residuals generated by the modified interval observer to increase the unknown disturbance robustness and the fault sensitivity, respectively. Finally, a numerical example is used to illustrate the effectiveness of the proposed fault detection scheme.

KEYWORDS: Fault detection; Interval observer; Discrete-time linear system

1 INTRODUCTION

With the industrial development, system is developing in the direction of large scale, complexity and high investment (Li et al., 2020, Shi et al., 2020, Wang et al., 2020, Shi et al., 2019, Li et al., 2019). Research on how to increase the safety of the system to reduce the probability of major accidents has become one of the urgent issues of modern industrial production system. At present, fault detection has been widely concerned since it can improve the reliability of the system. Among the existing fault detection scheme, observer-based fault detection scheme has developed rapidly, and the main idea are residuals generation and threshold setting. However, an incorrect selection of threshold may result in false alarms. There are many different threshold calculation schemes in the literature. For example, most of existing threshold selection methods depend on a constant (Wang et al., 2007), the dynamic threshold generator of linear system is derived (Johansson et al., 2006). Obviously, threshold selection is a problem worthy of further study.

Recently, the outstanding progress of the interval observer theory (Luenberger et al., 1964) and its application advantages in control have provided new ideas for fault detection. The so-called interval observer is to give the range of state change when there are unknown factors in the system. Using the interval observer as the residuals generator, the state interval can be seen as a natural threshold in fault detection. Interval observer can not only generate the residuals, but also give the natural threshold, omitting the threshold setting. The residuals interval is directly used for fault decision and the method is simple and intuitive. (Zhang et al., 2017) give a event-triggered fault detection scheme based on interval observer. (Zhang et al., 2019) presents a fault detection and isolation scheme based on modified interval observer for multi-agent systems. The figure 1 shows the principle of fault detection using interval observer.

![Fig 1. Fault detection scheme based on interval observer](image)

Using LMI technology to prove the stability and nonnegativity of the error system are the key problem of designing the interval observer. (Efimov et al., 2013) The coordinate transformation of the Sylvester equation to solve the above key problems. Based on this, (Efimov, W et al., 2013) used the Lyapunov stability theory to analyze the boundedness of the estimated error dynamics. In (Chebotarev et al., 2015), the robustness of the interval is analyzed in the $L_1/L_2$ framework, and the estimation accuracy is optimized.
In this paper, a modified interval observer is conceived based on the traditional Luenberger interval observer, which can make the residuals perform better. The residuals disturbance robustness and fault sensitivity are improved. In addition, the Lyapunov function is transferred to the slack variable matrix, which reduce the conservativeness of LMI stability condition.

The main contributions are the following:

1. A new interval observer is designed, which has more design freedom than the Luenberger interval observer.
2. By introducing the $l_1/H_\infty$ performance, the residuals generator by the modified interval observer has better unknown disturbance robustness and fault sensitivity.

**Notation:** $y^+$ is the generalized inverse of matrix $y$, $A^T$ represents the transposition of matrix $A$. For given $B$ define $B^+ = \max \{0, B\}$ and $B^- = B^+ - B$.

### 2 SYSTEM PRESENTATION AND PRELIMINARIES

#### 2.1 System presentation

The discrete-time linear system is considered.

\[ x(k+1) = Ax(k) + E\sigma(k) + Ff(k) \]

\[ y(k) = Cx(k) \]  \hspace{1cm} (1)

where $x(k) \in R^n$, $y(k) \in R^p$, $\sigma(k) \in R^p$, $f(k) \in R^m$ are state, measurement output, disturbance, fault signal. $A \in R^{n \times n}$, $E \in R^{n \times d}$, $F \in R^{n \times m}$, $C \in R^{p \times n}$.

**Assumption 1:** Matrix $A$ in discrete-time linear system (1) is Schur stable.

**Assumption 2:** Discrete-time system include disturbance $\sigma(k)$ are exist the known bound functions $\underline{\sigma}(k), \overline{\sigma}(k)$, such as $\underline{\sigma}(k) < \sigma(k) < \overline{\sigma}(k)$  \hspace{1cm} (2)

**Remark 1:** From (Hajshirmohamadi et al., 2016), Assumption 1 does not limit the observer’s design. Assumption 2 represents unknown disturbances range.

**Lemma 1:** Let $x(k) \in R^n$ and $\chi(k) < x(k) < \overline{\chi}(k)$ for some $\chi(k), \overline{\chi}(k) \in R^n$, then

\[ E^+ \underline{\sigma}(k) - E^- \overline{\sigma}(k) \leq E\sigma(k) \leq E^+ \overline{\sigma}(k) - E^- \underline{\sigma}(k) \]

\[ C^+ \chi(k) - C^- \overline{\chi}(k) \leq Cx(k) \leq C^+ \underline{\chi}(k) - C^- \chi(k) \]  \hspace{1cm} (3)

#### 2.2 Preliminaries

According to the discrete-time linear system (1), the modified interval observer is designed as follows:

\[ \begin{align*}
\dot{\xi}(k+1) &= TA\hat{x}(k) + \overline{L}(y(k) - C\hat{x}(k)) + \overline{\Delta} \\
\hat{x}(k) &= \xi(k) + Ny(k) \\
\overline{\chi}(k) &= \xi(k) \\
\overline{y}(k) &= C^+\chi(k) - C^-\chi(k) \\
\overline{r}(k) &= V(y(k) - \overline{\chi}(k)) \\
\xi(k+1) &= T\hat{A}\xi(k) + \underline{L}(y(k) - C\hat{x}(k)) + \underline{\Delta} \\
\underline{\chi}(k) &= \xi(k) + Ny(k) \\
\underline{y}(k) &= C^+\chi(k) - C^-\chi(k) \\
\underline{r}(k) &= V(y(k) - \underline{\chi}(k))
\end{align*} \]  \hspace{1cm} (4)

where $\overline{\xi}(k), \underline{\xi}(k), \hat{x}(k)$ and $\hat{\chi}(k)$ are intermediate variables, $\overline{\chi}(k), \underline{\chi}(k), \overline{\chi}(k), \overline{y}(k), \underline{r}(k), \underline{r}(k)$ denote the upper and lower estimates of state $x(k)$, disturbances, residuals, $\overline{L}$ and $L$ are gain matrices. $\overline{\Delta}$ and $\underline{\Delta}$ are as follows:

\[ \overline{\Delta} = (TE)^+\overline{\sigma}(k) - (TE)^-\underline{\sigma}(k) \]

\[ \underline{\Delta} = (TE)^+\underline{\sigma}(k) - (TE)^-\overline{\sigma}(k) \]

In (4), $T \in R^{n \times n} > 0$, $N \in R^{n \times m} > 0$ and the matrices $T$ and $N$ satisfying

\[ T + NC = I_n \]  \hspace{1cm} (5)

**Remark 2:** Compared with the single gain matrix $L$ of the Luenberger interval observer (Zhang et al., 2017). The proposed interval observer contains more gain matrices $T$ and $N$.

Define $e_\overline{\xi}(k) = \overline{\xi}(k) - Tx(k)$, $e_\underline{\xi}(k) = \underline{\xi}(k) - x(k)$, $e_\overline{\chi}(k) = \overline{\chi}(k) - \chi(k)$ and $e_\underline{\chi}(k) = \underline{\chi}(k) - \chi(k)$. Combining (1) and (4), we have

\[ \begin{align*}
e_\overline{\xi}(k+1) &= \overline{\xi}(k+1) - Tx(k+1) \\
&= (TE)^+\overline{\sigma}(k) - (TE)^-\underline{\sigma}(k) \\
&- TAx(k) - TE\overline{\sigma}(k) - \overline{TFF}(k) \\
&= (TA - \overline{L}C)e_\overline{\xi}(k) + TEF(\overline{\sigma}(k)) \\
&+ (TE)^+\overline{\sigma}(k) - (TE)^-\underline{\sigma}(k) \\
&- TFF(k)
\end{align*} \]  \hspace{1cm} (6)

According to (5), $e_\overline{\xi}(k)$ satisfies the following equation.

\[ \begin{align*}
e_\overline{\xi}(k) &= \overline{\xi}(k) + Ny(k) - x(k) \\
&= \overline{\xi}(k) + NCx(k) - x(k) \\
&= e_\overline{\xi}(k)
\end{align*} \]  \hspace{1cm} (7)

Substituting (7) into (6), we obtain

\[ \begin{align*}
e_\overline{\xi}(k+1) &= (TA - \overline{L}C)e_\overline{\xi}(k) - TE\overline{\sigma}(k) \\
&+ (TE)^+\overline{\sigma}(k) - (TE)^-\underline{\sigma}(k) \\
&- TFF(k)
\end{align*} \]  \hspace{1cm} (8)

Similarly, the following equation is obtained.

\[ \begin{align*}
e_\underline{\xi}(k+1) &= (TA - \overline{L}C)e_\underline{\xi}(k) + TEF(\underline{\sigma}(k)) \\
&- (TE)^+\overline{\sigma}(k) + (TE)^-\underline{\sigma}(k)
\end{align*} \]
According to (3), the following inequality is obtained.

\[(TE)^+ \bar{\omega}(k) - (TE)^- \omega(k) - TE \omega(k) \geq 0 \quad (10)\]

\[TE \omega(k) - (TE)^+ \bar{\omega}(k) + TE^- \bar{\omega}(k) \geq 0 \quad (11)\]

According to (10) and (11), when the matrices \((TA - \bar{L}C)\) and \((TA - LC)\) guarantees nonnegative and \(x(0) \leq x(0) \leq \bar{x}(0)\), the following relation is maintained in no fault case \((f(k) = 0)\).

(a) \(e^T(k) \geq 0 \) and \(e(k) \geq 0\);
(b) \(e^T(k) \geq 0 \) and \(e(k) \geq 0\);
(c) \(0 \in [\bar{r}(k), \bar{r}(k)]\).

If fault occur \((f(k) \neq 0)\), (a) cannot be maintained.

Denoting

\[e(k) = \begin{bmatrix} e^T(k) \\ e(k) \end{bmatrix}, \quad r(k) = \begin{bmatrix} \bar{r}(k) \\ r(k) \end{bmatrix} \quad \text{and} \quad \bar{\omega}(k) = \begin{bmatrix} \bar{\omega}(k) \\ \bar{\omega}(k) \end{bmatrix}.\]

The discrete-time system (8) and (9) are converted into the following form.

\[\epsilon(k + 1) = \tilde{A} \epsilon(k) + \tilde{E} \bar{\omega}(k) + \tilde{F} \bar{r}(k) \quad r(k) = C_1 \epsilon(k) + ye(k) \quad \text{and} \quad \bar{\omega}(k) = C_2 \epsilon(k) \quad (12)\]

where

\[\tilde{A} = \begin{bmatrix} TA - \bar{L}C & 0 & 0 \\ 0 & TA - L \mathcal{C} & 0 \\ 0 & 0 & A \end{bmatrix}, \quad \tilde{E} = \begin{bmatrix} (TE)^+ & (TE)^- & 0 \\ (TE)^- & (TE)^+ & 0 \\ 0 & 0 & \bar{E} \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} -TF \\ TF \end{bmatrix}, \]

\[C_1 = \begin{bmatrix} VC^- \\ -VC^+ \\ -VC^+ \\ -VC^+ \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.\]

The main idea of interval observer (4) to increase the residuals disturbance robustness and the fault sensitivity are designed as follows:

(I) \(TA - \bar{L}C\) and \(TA - L \mathcal{C}\) are nonnegative matrices and satisfy Schur stable.

(II) \((l_1\) performance) Conditions for the influence of unknown disturbance \(\bar{\omega}(k)\) on the residuals \(r(k)\) without fault:

\[
\sup \frac{\|r(k)\|_{\infty}}{\|\bar{\omega}(k)\|_{\infty}} < \alpha \quad (13)
\]

where \(\bar{\omega}(k) \in l_\infty\).

(III) \((H_\infty\) performance) Conditions for the influence of fault \(f(k)\) on the residuals \(r(k)\) without disturbance:

\[
\sup \frac{\|r(k) - jf(k)\|_2}{\|f(k)\|_2} < \beta \quad (14)
\]

where \(j\) is a weight matrix, \(f(k) \in l_2 \neq 0\).

Remark 3: (II) represents the residuals disturbance robustness conditions. (III) represents the condition for improve the residuals fault sensitivity.

The following lemmas are given to prepare for the following proof.

**Lemma 2:** Given the matrix \(x \in R^{a \times b}\), \(y \in R^{b \times c}\) and \(z \in R^{a \times c}\), \(\text{rank}(y) = c\), \(xy = z\). The results are as follows:

\[x = zy^+ + S(I - yy^+) \quad (15)\]

**Lemma 3:** The following two inequalities are equivalent

(I) The existence of symmetric matrix \(\Xi > 0\) makes \(A^T \Xi A + Z < 0\) \( (16)\)

(II) The existence of symmetric matrix \(\Xi\) and matrix \(G\) makes

\[\left[ \begin{array}{cc}
\Xi - ATG & -A^T \\
-GTA & \Xi - G - G^T
\end{array} \right] < 0 \quad (17)\]

**Lemma 4:** Given symmetric matrix \(S = [S_{11} \ S_{12}]\), \(\bar{S}_{12} \in \mathcal{R}^{r \times r}\). The following three conditions are equivalent.

(I) \(S < 0\)

(II) \(S_{11} < 0\), \(S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0\)

(III) \(S_{22} < 0\), \(S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0\)

### 3 FAULT DETECTION SCHEME

The constraint conditions of residuals disturbance robustness, residuals fault sensitivity and nonnegative are given, and finally transformed it into parameters convex optimization problems to obtain the optimal solution.

#### 3.1 Condition of disturbance attenuation

The residuals disturbance attenuation conditions are obtained.

**Theorem 1:** Consider the system (12) with \(f(k) = 0\), for \(l_1\) performance index \(\alpha > 0\). The existence matrix \(P_{1} > 0\), \(W\) satisfied the following inequality.

\[
\Theta < 0 \quad (18)
\]

\[
\Omega < 0 \quad (19)
\]

where

\[
\Theta = \begin{bmatrix}
\Theta_{11} & 0 & \Theta_{13} \\
0 & \Theta_{22} & \Theta_{23} \\
\Theta_{31} & \Theta_{32} & \Theta_{33}
\end{bmatrix}, \quad \Theta_{11} = -\lambda P_{1}, \quad \Theta_{13} = -A^T W, \quad \Theta_{22} = -\mu I, \quad \Theta_{23} = -E^T W, \quad \Theta_{33} = -\Omega_{13}^T \Omega_{13}
\]

\[P_{1} - W - W^T, \quad \Lambda = \begin{bmatrix}
\Omega_{11} & 0 \\
0 & \Omega_{22} \\
\Theta_{31} & \Theta_{32}
\end{bmatrix}, \quad \Omega_{11} = -(1 - \lambda) P_{1}, \quad \Omega_{13} = -C^T W, \quad \Omega_{22} = -(\alpha - \mu) I, \quad \Omega_{33} = \alpha^{-1} - W - W^T.
\]

**Proof:**

The following Lyapunov function is considered.

\[V_{1}(\epsilon(k)) = e^T(k) P_{1} e(k) \quad (20)\]
System (12) with \( f(k) = 0 \) satisfies the \( l_1 \) performance (13) when the following inequality are guaranteed.

\[
V_1(e(k+1)) - \nu V_1(e(k)) - \mu \tilde{\alpha}(k) < 0
\]  
(21)

\[
r^T(k) r(k) - \alpha[(1 - \lambda)V_1(e(k)) + (\alpha - \mu)\tilde{\alpha}(k)] < 0
\]  
(22)

where \( 0 < \lambda < 1, \mu > 0 \).

According to the Lyapunov function (20), inequality (21) and system (12) with \( f(k) = 0 \), the following equation is obtained.

\[
V_1(e(k+1)) - \nu V_1(e(k)) - \mu \tilde{\alpha}(k) = e^T(k)\tilde{T}_1 P_1 e(k) + 2e^T(k)\tilde{T}_2 P_1 \tilde{E} \tilde{\alpha}(k) + \tilde{\alpha}(k)\tilde{E} P_1 \tilde{E} \tilde{\alpha}(k) - \lambda e^T(k)P_1 e(k)
\]  
where

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\]  

(23)

According to equation (23), inequality (21) is transformed into the following inequality.

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(24)

Inequality (24) is transformed into the following form to prepare for the subsequent introduction of slack variable \( W \).

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(25)

According to Lemma 3, the slack variable is introduced into (25).

\[
\begin{bmatrix}
-\lambda \nu & 0 & -\mu I
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(26)

According to inequality (22) and system (12), the following form is obtained.

\[
\begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(27)

We get

\[
\begin{bmatrix}
-\lambda \nu & 0 & -\mu I
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(28)

Inequality (28) is converted into the following form.

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(29)

According to Lemma 2, the slack variable \( W \) is introduced into (29).

\[
\begin{bmatrix}
1 & 0 & -\tilde{T}_1 W \\
* & -\alpha I & 0 \\
* & * & \alpha^{-1} - W - W^T
\end{bmatrix}
< 0
\]  
(30)

**Remark 4:** Inequalities (26) and (30) satisfy the conditions of disturbance attenuation.

### 3.2 Fault sensitivity condition

The fault signal sensitivity conditions are given.

**Theorem 2:** The system (12) with \( \tilde{\alpha}(k) = 0 \) is considered, if the existence matrix \( P_2 \) and \( W \) makes the following form.

\[
\Lambda = \begin{bmatrix}
\Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\
\Lambda_{31} & \Lambda_{32} & \Lambda_{33} & 0 \\
\Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44}
\end{bmatrix}
\]  
(31)

where \( \Lambda_{11} = -P_2 \).

\[\Lambda_{13} = C_1^T, \quad \Lambda_{14} = -\tilde{W}, \quad \Lambda_{22} = \beta^2 I, \quad \Lambda_{23} = -J^T, \quad \Lambda_{24} = -\tilde{F}^T W, \quad \Lambda_{33} = -I, \quad \Lambda_{44} = P_2 - W - W^T. \]

**Proof:**

The Lyapunov function is defined in the following form.

\[
V_2(e(k)) = e^T(k)P_2 e(k)
\]  
(32)

The following inequality will satisfy \( H_{\infty} \) performance (14).

\[
V_2(e(k+1)) - V_2(e(k)) + r(k) - Jf(k)^T r(k) - Jf(k) - \beta^2 f^T f(k) < 0
\]  
(33)

According to (32) and (33), the following equation is obtained.

\[
V_2(e(k+1)) = V_2(e(k)) + r(k) - Jf(k)^T r(k) - Jf(k) - \beta^2 f^T f(k)
\]  
(34)

According to inequality (22) and system (12), the following form is obtained.

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(25)

We get

\[
\begin{bmatrix}
\Lambda_{11} & 0 & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\
0 & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\
\Lambda_{31} & \Lambda_{32} & \Lambda_{33} & 0 \\
\Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44}
\end{bmatrix}
< 0
\]  
(26)

Inequality (28) is converted into the following form.

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(27)

We get

\[
\begin{bmatrix}
\Lambda_{11} & 0 & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\
0 & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\
\Lambda_{31} & \Lambda_{32} & \Lambda_{33} & 0 \\
\Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44}
\end{bmatrix}
< 0
\]  
(28)

Inequality (28) is converted into the following form.

\[
\begin{bmatrix}
\tilde{T}_1 & \tilde{T}_2
\end{bmatrix}
\begin{bmatrix}
P_1 & \tilde{E}
\end{bmatrix}
\begin{bmatrix}
in(e(k)) \\
\tilde{\alpha}(k)
\end{bmatrix}
< 0
\]  
(29)

According to Lemma 2, the slack variable \( W \) is introduced into (29).

\[
\begin{bmatrix}
1 & 0 & -\tilde{T}_1 W \\
* & -\alpha I & 0 \\
* & * & \alpha^{-1} - W - W^T
\end{bmatrix}
< 0
\]  
(30)
By using Lemma 4, we can rewrite the above inequality as
\[
\begin{bmatrix}
\bar{A}^T P_2 \bar{A} - P_2 & \bar{A}^T P_2 \bar{F} - \beta^2 I \\
* & \bar{F}^T P_2 \bar{F} - \beta^2 I - I^T
\end{bmatrix} < 0
\] (36)
It also can be rewritten as
\[
\begin{bmatrix}
\bar{A}^T \\
\bar{F}^T
\end{bmatrix}
\begin{bmatrix}
P_2[\bar{A} \bar{F} 0] & * & -\beta^2 I & -I^T
\end{bmatrix} < 0
\] (37)

According to Lemma 3, the slack variable \( W \) is introduced into (36).
\[
\begin{bmatrix}
-P_2 & 0 & C^T \\
* & -\beta^2 I & -I^T \\
* & * & 0
\end{bmatrix} < 0
\] (38)

Remark 5: Inequalities (37) satisfy the conditions of fault sensitivity.

### 3.3 Nonnegative condition

The nonnegativity conditions of the matrices \( TA - \bar{L}C \) and \( TA - \bar{C}L \) are presented based on LMIs. The following system example is used to illustrate the LMI solution of nonnegativity.

\[
\begin{align*}
A & = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \\
\bar{L} & = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}, \\
\bar{C} & = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \\
\bar{C}^T W & = \begin{bmatrix} -\ell_1 c_1 \\ -\ell_2 c_2 \end{bmatrix}, \\

T & = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \\
C & = \begin{bmatrix} c_1 & c_2 \end{bmatrix},
\end{align*}
\]

then
\[
\begin{align*}
TA - \bar{L}C & = \begin{bmatrix} t_{11} a_{11} + t_{12} a_{21} - \ell_1 c_1 & t_{11} a_{12} + t_{12} a_{22} - \ell_1 c_2 \\ t_{21} a_{11} + t_{22} a_{21} - \ell_2 c_1 & t_{21} a_{12} + t_{22} a_{22} - \ell_2 c_2 \end{bmatrix}, \\

TA - \bar{C}L & = \begin{bmatrix} t_{11} a_{11} + t_{12} a_{21} - \ell_1 c_1 & t_{11} a_{12} + t_{12} a_{22} - \ell_1 c_2 \\ t_{21} a_{11} + t_{22} a_{21} - \ell_2 c_1 & t_{21} a_{12} + t_{22} a_{22} - \ell_2 c_2 \end{bmatrix}.
\end{align*}
\] (39)

According to the (39), the nonnegative constraint as follows:
\[
t_{ij} a_{ij} - \ell_i c_i \geq 0, \\
t_{ij} a_{ij} - \ell_i c_i \geq 0, i,j = 1,2
\] (40)

By introducing the slack variable \( W \), the following equations are obtained.
\[
X = \bar{L}^T W, \quad Y = \bar{C}^T W
\] (41)
where \( X \) and \( Y \) unknown variables.
\[
X^T = W \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} W \ell_1 \\ W \ell_2 \end{bmatrix} = [X_1, X_2]^T.
\]

Remark 6: Nonnegativity can be obtained by using the same principle for any order system.

### 3.4 Fault detection decision

The following algorithm is used to obtained the optimal solution of interval observer (4) gain.

**Algorithm 1**: Fault detection decision

1. Compute modified interval observer gains: By solving the optimization problem below presented.
\[
\min \rho_1 \alpha + \rho_2 \beta \\
s.t. (26), (30), (38)
\]

2. Fault detection decision: If \( 0 \notin [r(k), \bar{r}(k)] \) alarm.

Remark 7: The fault detection scheme given in this paper is obviously different from the existing method in (Leao et al., 2019, Nemati et al., 2019, Khan et al., 2011, Aouaouda et al., 2015). Its advantages of fault detection based interval observer is that the interval observer can omitting to design residual generators and threshold selectors.

### 4 SIMULATION EXAMPLE

We use the numerical results to show the effectiveness of fault detection scheme.

The second-order discrete-time linear system with fault signal is considered.
\[
x(k+1) = Ax(k) + E \sigma(k) + Ff(k) \\
y(k) = Cx(k)
\] (44)
where \( A = \begin{bmatrix} 0.3 & -0.7 \\ 0.6 & -0.5 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \sigma(k) = 0.1 \sin(2 \pi k) + \text{rand}(-0.1,0.1), \) sample period \( T = 1s \).

The parameters are chosen as \( \lambda = 0.3, \mu = 0.4 \). We set \( \rho_1 = \rho_2 = j = 1 \). We set \( \rho_1 = \rho_2 = j = 1 \). We set \( \rho_1 = \rho_2 = j = 1 \). We set \( \rho_1 = \rho_2 = j = 1 \). We set \( \rho_1 = \rho_2 = j = 1 \).

We use the YALMIP toolbox in MATLAB and Lemma 2. The gain matrices in the interval observer (4) are obtained as
\[
\begin{align*}
\bar{L} & = \begin{bmatrix} -0.4303 \\ -0.0006 \end{bmatrix}, T = \begin{bmatrix} 1 & -0.4702 \\ 0 & 0.274 \end{bmatrix}, \\
N & = \begin{bmatrix} 0.4702 \\ 0.9726 \end{bmatrix}.
\end{align*}
\]

The initial conditions are set as \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). We set \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). We set \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). We set \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). We set \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). We set \( \bar{x}(0) = \bar{x}(0) = x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).
Fig 2. State $x_1(k)$ and state $x_1(k)$ estimations $\bar{x}_1(k)$, $\bar{x}_1(k)$ without fault.

Fig 3. Upper residuals $\bar{r}(k)$ and lower residuals $r(k)$ without fault.

Fig 4. State $x_1(k)$ and state $x_1(k)$ estimations $\bar{x}_1(k)$, $\bar{x}_1(k)$ with fault.
When no fault occurs, figure 2 and figure 3 are obtained. Figure 2 shows that $x_3(k) < x_1(k) < \bar{x}_1(k)$, that is state $x_1(k)$ always in the middle of upper state $\bar{x}_1(k)$ and lower $x_3(k)$. Figure 3 shows that the upper and lower residual contain the origin. According to algorithm 1, no fault occurs at this time.

The following constant fault signal is considered.

$$f(k) = \begin{cases} 1,2, & k \geq 75 \\ 0, & otherwise \end{cases}$$ (45)

When fault occurs, figures 4 and 5 are obtained. Figure 4 shows when $k \geq 75$, state $x_3(k)$ is not in the middle of upper state $\bar{x}_1(k)$ and lower $x_3(k)$. Figure 5 shows when $k \geq 75$, the upper and lower residual does not contain the origin. According to algorithm 1, fault occurs at this time.

Through the simulation results, we obtain that the fault detection scheme based on modified interval observer proposed for discrete-time linear systems in this paper can quickly and accurately detect whether the system fault.

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7 REFERENCES


